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ON LOCAL LINEAR FUNCTIONALS FOR L-SPLINES.(U)
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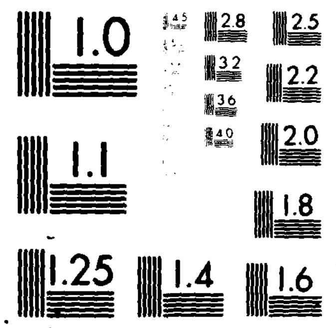

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ON LOCAL LINEAR FUNCTIONALS FOR L-SPLINES

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(6) ON LOCAL LINEAR FUNCTIONALS FOR L-SPLINES.

(10) Rong-Qing Jia* (12) 18

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ABSTRACT

Quasi-interpolant functionals for L-splines are constructed. With them as a tool, an explicit construction of LB-splines is done, and a quick proof of the existence and uniqueness of the expansion of an L-spline in an LB-spline series is given. Moreover, a necessary and sufficient condition for a function, under which it generates a local linear functional that vanishes at all LB-splines but one, is obtained.

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Key Words: LB-splines, local, linear functional

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SIGNIFICANCE AND EXPLANATION

B-splines play an important role in spline function theory. One is deeply impressed by the effect of quasi-interpolant functionals in B-spline theory. With them as a tool, some problems become easier to solve, and some important results are obtained. When one deals more generally with L-splines, that is, splines associated with a linear differential operator, an attempt to construct similar functionals for LB-splines naturally arises, and there is reason to claim that such functionals would be helpful for studying L-splines.

In the present report, such a construction of quasi-interpolant functionals and local linear functionals is carried out.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ON LOCAL LINEAR FUNCTIONALS FOR L-SPLINES

Rong-Qing Jia*

§1. INTRODUCTION

We begin with some notations and definitions.

Let $k \in \mathbb{N}$, $\underline{\underline{t}} := (t_i)$ nondecreasing (finite, infinite or biinfinite) with $t_i < t_{i+k}$, all i, and let

$$a := \inf\{t_i\},\ b := \sup\{t_i\}\ ,$$
$$c_i := \max\{m; t_{i-m} = t_i\}\ ,$$
$$\ell_i := \max\{m; t_{i+m} = t_i\}\ ,$$
$$d_i := c_i + \ell_i + 1\ ,$$
$$\text{jump}_{t_i} f := f(t_i+) - f(t_i-)\ .$$

Let $H_p^k(a,b)$ denote the space of functions which are k-fold integrals of functions in $L_p(a,b)$, $1 \le p \le \infty$. Further, let

$$L = \sum_{j=0}^{k} p_j D^{k-j}$$

be a nonsingular k-th order differential operator, where $p_0 \equiv 1$, $p_j \in C^j(a,b)$ $(j = 1,\dots,k)$ and $D = \frac{d}{dx}$. Then the formal adjoint operator of L is

$$L^* = \sum_{j=0}^{k} (-1)^j D^j (p_{k-j}\cdot)\ .$$

By N_L and N_{L^*} we denote the null spaces of L and L^*, respectively.

Throughout this paper the following condition:

(ET) "The sum of multiplicities of g's zeros does not exceed $k-1$ for any nonzero $g \in N_{L^*}$ and any i"

is supposed to hold.

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Definition 1.1. A function S defined on (a,b) is called an L-spline with knots $\underline{\underline{t}}$ if

(i) $S|_{(t_i,t_{i+1})} \in N_L|_{(t_i,t_{i+1})}$ for all i;

(ii) $\text{jump}_{t_i} S^{(\gamma)} = 0$ for all i and $\gamma < k - d_i$.

Definition 1.2. $[i,j]$ is called the carrier of the L-spline S if

(i) $S = 0$ outside $[t_i,t_j]$;

(ii) $\text{jump}_{t_i} S^{(\gamma)} = 0$ for $\gamma < k - \ell_i - 1$, but $\text{jump}_{t_i} S^{(k-\ell_i-1)} \neq 0$;

(iii) $\text{jump}_{t_j} S^{(\gamma)} = 0$ for $\gamma < k - c_j - 1$, but $\text{jump}_{t_j} S^{(k-c_j-1)} \neq 0$.

Definition 1.3. A nonzero L-spline with minimum carrier is called an LB-spline.

The purpose of this paper is to extend some results of polynomial B-splines to LB-splines. In §2 we construct quasi-interpolant functionals for LB-splines. In §3 we give an explicit construction of LB-splines. In §4 we obtain the expansion of an L-spline in an LB-spline series with the quasi-interpolant functionals as a tool. In §5 we extend de Boor's results about local linear functionals to LB-splines.

§2. QUASI-INTERPOLANT

For a fixed integer i, let μ_m be the functional given by

$$\mu_m(f) = \begin{cases} f^{(m-i-1)}(t_m) & \text{when } m = i+1,\ldots,i+\ell_i\,; \\ f^{(\ell_m)}(t_m) & \text{when } m \geq i+\ell_i+1\,. \end{cases} \tag{2.1}$$

Lemma 2.1. There exists a non-zero function $u_i(x) \in N_{L^*}$ which satisfies

$$\mu_m(u_i) = 0,\ m = i+1,\ldots,i+k-1.$$

Moreover, such a function is unique up to a constant factor.

Proof. Let $\varphi_1, \varphi_2, \ldots, \varphi_k$ be a basis of N_{L^*}. It is easily seen that the function

$$u_i(x) = \begin{vmatrix} \mu_{i+1}(\varphi_1) & \mu_{i+2}(\varphi_1) & \cdots & \mu_{i+k-1}(\varphi_1) & \varphi_1(x) \\ \mu_{i+1}(\varphi_2) & \mu_{i+2}(\varphi_2) & \cdots & \mu_{i+k-1}(\varphi_2) & \varphi_2(x) \\ \vdots & \vdots & & \vdots & \vdots \\ \mu_{i+1}(\varphi_k) & \mu_{i+2}(\varphi_k) & \cdots & \mu_{i+k-1}(\varphi_k) & \varphi_k(x) \end{vmatrix} \tag{2.2}$$

satisfies

$$\mu_m(u_i) = 0,\ m = i+1,\ldots,i+k-1\,.$$

We claim that

$$u_i(x) \neq 0 \quad \text{when} \quad x \in (t_j, t_{j+1}),\quad j = i,\ldots,i+k-1\,.$$

Suppose to the contrary that there exists some $x \in (t_j, t_{j+1})$ $(j = i,\ldots,i+k-1)$ for which $u_i(x) = 0$. Then we can find $\gamma_1, \gamma_2, \ldots, \gamma_k$, of which at least one is not zero, so that

$$\gamma_1\mu_j(\varphi_1) + \gamma_2\mu_j(\varphi_2) + \ldots + \gamma_k\mu_j(\varphi_k) = 0,\quad j = i+1,\ldots,i+k-1$$

and

$$\gamma_1\varphi_1(x) + \gamma_2\varphi_2(x) + \ldots + \gamma_k\varphi_k(x) = 0\,.$$

Let $\varphi = \gamma_1\varphi_1 + \gamma_2\varphi_2 + \ldots + \gamma_k\varphi_k$. Then φ is not a zero function, and the sum of the multiplicities of φ's zeros exceeds $k-1$. This contradicts the condition (ET).

Suppose now that another function v has the same property as u_i. We have to show that there exists a constant c such that $v = cu_i$. There are the following two possibilities:

(i) $t_i < t_{i+1}$. In this case it follows from the condition (ET) that $u_i(t_i) \neq 0$ and $v(t_i) \neq 0$. If we put $c = v(t_i)/u_i(t_i)$, then the function $v - cu_i \in N_{L^*}$ and the sum of multiplicities of its zeros would exceed or equal k, hence $v - cu_i = 0$, that is, $v = cu_i$.

(ii) $t_i = t_{i+1}$. Thus we know that $u_i^{(\ell_i)}(t_i) \neq 0$ and $v^{(\ell_i)}(t_i) \neq 0$ in view of the condition (ET). A similar demonstration gives that $v = cu_i$ for $c = v^{(\ell_i)}(t_i)/u_i^{(\ell_i)}(t_i)$.

The determinant on the right-hand side of (2.2) is abbreviated to

$$\det\begin{pmatrix} \mu_{i+1}, \mu_{i+2}, \ldots, \mu_{i+k-1}, x \\ \varphi_1, \varphi_2, \ldots, \varphi_{k-1}, \varphi_k \end{pmatrix} .$$

Corollary 2.1. If $\psi_1, \psi_2, \ldots, \psi_k$ is another basis of N_{L^*}, then there exists a constant c such that

$$\det\begin{pmatrix} \mu_{i+1}, \mu_{i+2}, \ldots, \mu_{i+k-1}, x \\ \psi_1, \psi_2, \ldots, \psi_{k-1}, \psi_k \end{pmatrix} \equiv c \cdot \det\begin{pmatrix} \mu_{i+1}, \mu_{i+2}, \ldots, \mu_{i+k-1}, x \\ \varphi_1, \varphi_2, \ldots, \varphi_{k-1}, \varphi_k \end{pmatrix} . \tag{2.3}$$

Now we consider Lagrange's Formula [7]. If $f \in H_p^k(\alpha,\beta)$ and $g \in H_q^k(\alpha,\beta)$, where $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_\alpha^\beta (Lf)g\,dx = \int_\alpha^\beta (L^*g)f\,dx + W(f,g;x)\Big|_\alpha^\beta \tag{2.4}$$

where

$$W(f,g;x) = \sum_{\gamma=0}^{k} \{f^{(\gamma-1)}(x)[p_{k-\gamma}(x)g(x)] - f^{(\gamma-2)}(x)[p_{k-\gamma}(x)g(x)]' + \ldots + (-1)^{\gamma-1} f(x)[p_{k-\gamma}(x)g(x)]^{(\gamma-1)}\} . \tag{2.5}$$

In particular, if $f\big|_{(\alpha,\beta)} \in N_L$ and $g\big|_{(\alpha,\beta)} \in N_{L^*}$, then it follows from (2.4) that

$$W(f,g;\alpha+) = W(f,g;\beta-) . \tag{2.6}$$

Taking an L-spline S as f and taking u_i as g in (2.5), we have

$$W(S,u_i;x) = \sum_{\gamma=0}^{k} \{S^{(\gamma-1)}(x)[p_{k-\gamma}(x)u_i(x)] - S^{(\gamma-2)}(x)[p_{k-\gamma}(x)u_i(x)]' + \dots + (-1)^{\gamma-1}S(x)[p_{k-\gamma}(x)u_i(x)]^{(\gamma-1)}\} . \tag{2.7}$$

If $t_i < t_m < t_{i+k}$, then

$$u_i(t_m) = \dots = u_i^{(d_m-1)}(t_m) = 0 ,$$

$$\text{jump}_{t_m} S = \dots = \text{jump}_{t_m} S^{(k-d_m-1)} = 0 ,$$

hence

$$W(S,u_i;t_m+) = W(S,u_i;t_m-) .$$

On the other hand, we have, for any $\xi,\eta \in (t_i,t_{i+k})$,

$$W(S,u_i;\eta) - W(S,u_i;\xi) = \sum_{\xi \le t_m < \eta} [W(S,u_i;t_m+) - W(S,u_i;t_m-)] .$$

Therefore,

$$W(S,u_i;\eta) - W(S,u_i;\xi) = 0 ,$$

that is,

$$W(S,u_i;\eta) = W(S,u_i;\xi), \quad \text{for any } \xi,\eta \in (t_i,t_{i+k}) . \tag{2.8}$$

We conclude that $W(S,u_i;\cdot)$ is identically equal to a constant in (t_i,t_{i+k}).

Definition 2.1. By $\mathcal{L}(L;\underline{t})$ we denote the space of all L-splines with knots $\underline{t}$. The linear functional

$$\lambda_i : S \mapsto W(S,u_i;\xi), \quad t_i < \xi < t_{i+k} . \tag{2.9}$$

which acts on the space $\mathcal{L}(L;\underline{t})$ is called a quasi-interpolant functional.

Theorem 2.1. If S is an L-spline with $[m,n]$ as its carrier, then

(1°) $\lambda_i S = 0$ when $m > i$;

(2°) $\lambda_i S \neq 0$ when $m = i$;

(3°) $\lambda_i S = 0$ when $n < i + k$;

(4°) $\lambda_i S \neq 0$ when $n = i + k$.

Proof. (1°) If $t_m > t_i$, we take $\xi \in (t_i,t_m)$, then

$$\lambda_i S = W(S,u_i;\xi) = 0$$

since $S \equiv 0$ on (t_i, t_m). In the case of $t_m = t_i$, from

$$S(t_i) = S'(t_i) = \dots = S^{(k-\ell_i-1)}(t_i) = 0$$

$$u_i(t_i) = u_i'(t_i) = \dots = u_i^{(\ell_i-1)}(t_i) = 0$$

it follows that

$$\lambda_i S = W(S, u_i; t_i+) = 0 \ .$$

(2°) Suppose the converse statement $\lambda_i S = 0$ holds. There are two cases:

(i) $t_i < t_{i+1}$. Substituting $W(S, u_i; t_i+) = 0$ and

$$S(t_i) = S'(t_i) = \dots = S^{(k-2)}(t_i) = 0$$

into (2.7), we obtain

$$S^{(k-1)}(t_i+)u_i(t_i) = 0 \ ,$$

but $u_i(t_i) \neq 0$ in terms of the condition (ET) and $S^{(k-1)}(t_i+) \neq 0$, so we get a contradiction.

(ii) $t_i = t_{i+1}$. In this case,

$$S(t_i) = S'(t_i) = \dots = S^{(k-\ell_i-2)}(t_i) = 0 \ ,$$

$$u_i(t_i) = u_i'(t_i) = \dots = u_i^{(\ell_i-1)}(t_i) = 0 \ .$$

Combining it with (2.7), we have

$$S^{(k-\ell_i-1)}(t_i+)u_i^{(\ell_i)}(t_i) = 0 \ ,$$

which contradicts the fact that $S^{(k-\ell_i-1)}(t_i) \neq 0$ and $u_i^{(\ell_i)}(t_i) \neq 0$.

We can similarly prove (3°) and (4°).

Definition 2.2. If an L-spline S has $[m,n]$ as its carrier, then $n-m$ is called the length of S.

Corollary 2.2. The length of any nonzero L-spline S is at least k.

In fact, if $[m,n]$ is the carrier of S and $n - m < k$, then (2°) of Theorem 2.1 implies $\lambda_m S \neq 0$, but (3°) implies $\lambda_m S = 0$.

§3. THE CONSTRUCTION OF LB-SPLINES

There are other papers which deal with the construction of LB-splines (cf. Jerome and Schumaker [5]), but the construction given here is particularly suited for the development of the quasi-interpolant functionals. Further, we emphasize that LB-splines are entirely determined by the operator L and are independent of the choice of N_L's basis.

Lemma 3.1. If $\{\varphi_1,\varphi_2,\ldots,\varphi_k\}$ is a basis in N_L^*, then there exists a basis $\{\chi_1,\chi_2,\ldots,\chi_k\}$ in N_L such that, for $\ell = 0,1,\ldots,j$,

$$\sum_{i=1}^{k} \varphi_i^{(\ell)}(\xi)\chi_i^{(j-\ell)}(\xi) = \begin{cases} 0 & \text{when } j = 0,1,\ldots,k-2 ; \\ (-1)^{\ell} & \text{when } j = k-1 . \end{cases} \tag{3.1}$$

The functions (χ_i) are the adjunct functions for the (φ_i); see [6; 669]. Let

$$G(x,\xi) = \begin{cases} \sum_{i=1}^{k} \varphi_i(\xi)\chi_i(x), & x \geq \xi , \\ 0 & , x < \xi . \end{cases} \tag{3.2}$$

Clearly, $G(x,\xi)$ is Green's function for the operator L with side conditions:

$$y(\alpha) = y'(\alpha) = \ldots = y^{(k-1)}(\alpha) = 0,\ \alpha \leq x,\xi .$$

Now we define functionals ν_m as follows:

$$\nu_m(f) := \begin{cases} f^{(m-i)}(t_m), & m = i,\ldots,i+\ell_i ; \\ f^{(c_m)}(t_m), & m \geq i+\ell_i+1 . \end{cases} \tag{3.3}$$

It is easily seen that

$$K_m(x) = \nu_m(G(x,\cdot)),\ m = i,i+1,\ldots$$

are L-splines. By (3.1) we have

(i) For $m = i,\ldots,i+\ell_i$,

$$\text{jump}_{t_m} K_m^{(\gamma)} = \begin{cases} 0 & , \gamma < k-1-m+i ; \\ (-1)^{m-i} & , \gamma = k-1-m+i . \end{cases}$$

(ii) For $m \geq i+\ell_i+1$,

$$\text{jump}_{t_m} K_m^{(\gamma)} = \begin{cases} 0 & , \gamma < k-1-c_m ; \\ (-1)^{c_m} & , \gamma = k-1-c_m . \end{cases}$$

Thus the function

$$M_i(\varphi_1,\ldots,\varphi_k;x) := \begin{vmatrix} \nu_i(\varphi_1) & \nu_i(\varphi_2) & \cdots & \nu_i(\varphi_k) & \nu_i(G(x,\cdot)) \\ \nu_{i+1}(\varphi_1) & \nu_{i+1}(\varphi_2) & \cdots & \nu_{i+1}(\varphi_k) & \nu_{i+1}(G(x,\cdot)) \\ \vdots & \vdots & & & \\ \nu_{i+k}(\varphi_1) & \nu_{i+k}(\varphi_2) & \cdots & \nu_{i+k}(\varphi_k) & \nu_{i+k}(G(x,\cdot)) \end{vmatrix} \tag{3.4}$$

is an L-spline with $[i,i+k]$ as its carrier. The M_i's length equals k, but by Corollary 2.2 the length of any nonzero L-spline is not less than k, so we have already proved the main part of the following theorem.

Theorem 3.1. $M_i(\varphi_1,\varphi_2,\ldots,\varphi_k;x)$ given by (3.4) is an LB-spline. Moreover each LB-spline M can be represented as

$$M = \text{const}\cdot M_i(\varphi_1,\ldots,\varphi_k;\cdot) \quad \text{for some } i.$$

Proof. Suppose M's carrier is $[i,j]$. By Corollary 2.2 we know $j \geq i+k$, on the other hand, we have $j-i \leq k$ by the definition of LB-splines, so $j = i+k$. By Definition 1.2,

$$\text{jump}_{t_i} M_i^{(k-\ell_i-1)} \neq 0 \quad \text{and} \quad \text{jump}_{t_i} M^{(k-\ell_i-1)} \neq 0 .$$

Let

$$c := \text{jump}_{t_i} M^{(k-\ell_i-1)} / \text{jump}_{t_i} M^{(k-\ell_i-1)} .$$

Then $M - cM_i$ would have a carrier which is a proper subset of $[i,j]$. Applying Corollary 2.2 again to this case, we have $M - cM_i = 0$, that is, $M = cM_i$.

Corollary 3.1. For any two bases of $N_{L^*} = \{\varphi_1,\varphi_2,\ldots,\varphi_k\}$ and $\{\psi_1,\psi_2,\ldots,\psi_k\}$, there exists a nonzero constant c such that

$$M_i(\psi_1,\psi_2,\ldots,\psi_k;x) \equiv c\cdot M_i(\varphi_1,\varphi_2,\ldots,\varphi_k;x) .$$

§4. LB-SPLINES SERIES

It follows directly from Theorem 2.1 that

Theorem 4.1. For i, j integers, let M_j be an LB-spline with $[t_j, t_{j+k}]$ as its carrier, and let λ_i be a quasi-interpolant functional given by (2.9). Then

$$\lambda_i M_j \neq 0$$

if and only if $i = j$.

Corollary 4.1. For any open set I, $\{M_i;\ \mathrm{supp}\, M_i \cap I \neq \phi\}$ is linearly independent on I.

Proof. Suppose

$$\sum_{\mathrm{supp} M_i \cap I \neq \phi} \gamma_i M_i\big|_I = 0 .$$

Letting the functional $\lambda_i = W(\cdot, u_i; \xi_i)$ where $\xi_i \in \mathrm{supp}\, M_i \cap I$ act on the foregoing equation, we obtain

$$\gamma_i = 0 \quad \text{for all } i \text{ such that } \mathrm{supp}\, M_i \cap I \neq \phi .$$

Corollary 4.2. $\overline{\mathrm{supp}(\sum_i \gamma_i M_i)} = \overline{\bigcup_{\gamma_i \neq 0} \mathrm{supp}\, M_i}$

Proof. The relation

$$\overline{\mathrm{supp} \sum_i \gamma_i M_i} \subset \overline{\bigcup_{\gamma_i \neq 0} \mathrm{supp}\, M_i}$$

is obvious. Conversely, suppose $\tau \in \overline{\mathrm{supp}\, M_i}$ for some $i, \gamma_i \neq 0$, but $\tau \notin \overline{\mathrm{supp} \sum_i \gamma_i M_i}$. Then we can choose some τ_i inside $\mathrm{supp}\, M_i$ so that $\tau_i \notin \mathrm{supp} \sum_i \gamma_i M_i$. If we put $\lambda_i = W(\cdot, u_i; \tau_i)$, then

$$\lambda_i (\sum_i \gamma_i M_i) = 0 ,$$

hence $\gamma_i = 0$, which is a contradiction.

With the help of quasi-interpolant functionals we can obtain the following existence and uniqueness theorem about LB-spline series expansion. The proof is omitted here because it is similar to the proof in [3].

Theorem 4.2. Any L-spline S can be respresented as a series of LB-splines:

$$S = \sum_i \alpha_i M_i ;$$

moreover, this representation is unique.

§5. LOCAL LINEAR FUNCTIONALS

<u>Definition 5.1</u>. If

$$f^{(c_m)}(t_m) = g^{(c_m)}(t_m), \quad \forall_m , \tag{5.1}$$

then we say that f "agrees with" g at $\underline{\underline{t}}$ and write

$$f|_{\underline{\underline{t}}} = g|_{\underline{\underline{t}}} .$$

Suppose, for i integers, M_i are LB-splines, and u_i are given by (2.2). Let $n := i + k - c_{i+k}$. Then

$$t_i \le t_{n-1} < t_n = \cdots = t_{i+k} .$$

Let

$$u_i^+ = \begin{cases} 0 , & \text{if } t < (t_{n-1} + t_n)/2 ; \\ u_i , & \text{if } t \ge (t_{n-1} + t_n)/2 . \end{cases} \tag{5.2}$$

We have

<u>*Theorem 5.1*</u>. $h_i \in L_q(a,b)$ satisfies

$$\int h_i M_j = \delta_{ij}, \quad \text{all } i,j ,$$

if and only if $h_i = -L^* f$ for some $f \in H_q^k(a,b)$ with $f|_{\underline{\underline{t}}} = u_i^+|_{\underline{\underline{t}}}$.

<u>Proof</u>. "If" part. Suppose $f|_{\underline{\underline{t}}} = u_i^+|_{\underline{\underline{t}}}$. We have, for any L-spline S,

$$W(S,f;t_m+) = W(S,f;t_m-), \qquad m \le n - 1 , \tag{5.3}$$

and

$$W(S,f - u_i;t_m+) = W(S,f - u_i;t_m-), \qquad m \ge n . \tag{5.4}$$

In view of Lagrange's formula we have

$$\int_{t_m}^{t_{m+1}} (L^* f) S \, dx = \int_{t_m}^{t_{m+1}} (LS) f \, dx - W(S,f;x)\Big|_{t_m+}^{t_{m+1}-}$$

$$= W(S,f;t_m+) - W(S,f;t_{m+1}-), \qquad t_m < t_{m+1} ,$$

hence

$$\int (L^* f) M_j dx = \sum_{t_j \le t_m < t_{m+1} \le t_{j+k}} [W(M_j,f;t_m+) - W(M_j,f;t_{m+1}-)]. \tag{5.5}$$

Let us separate consideration of the following three possibilities.

(i) $t_{j+k} \le t_{n-1}$. In this case, it follows from (5.3) and (5.5) that

$$\int (L^*f)M_j dx = W(M_j,f;t_j+) - W(M_j,f;t_{j+k}-) ,$$

but

$$W(M_j,f;t_j+) = 0, \quad W(M_j,f;t_{j+k}-) = 0 \tag{5.6}$$

by (2.5) and the definition of LB-splines, so that $\int (L^*f)M_j dx = 0$.

(ii) $t_j \ge t_n$. We have, similarly,

$$W(M_j,f - u_i;t_j+) = 0, \quad W(M_j,f - u_i;t_{j+k}-) = 0 . \tag{5.7}$$

We rewrite (5.5) as

$$\int (L^*f)M_j dx = \sum_{t_j \le t_m < t_{m+1} \le t_{j+k}} [W(M_j,f - u_i;t_m+) - W(M_j,f - u_i;t_{m+1}-)]$$

$$+ \sum_{t_j \le t_m < t_{m+1} \le t_{j+k}} [W(M_j,u_i;t_m+) - W(M_j,u_i;t_{m+1}-)] .$$

The first sum is equal to zero by (5.4) and (5.7). To calculate the second sum we resort to Lagrange's Formula and obtain

$$\sum_{t_j \le t_m < t_{m+1} \le t_{j+k}} [W(M_j,u_i;t_m+) - W(M_j,u_i;t_{m+1}-)]$$

$$= \sum_{t_j \le t_m < t_{m+1} \le t_{j+k}} \left[\int_{t_m}^{t_{m+1}} (L^*u_i)M_j dx - \int_{t_m}^{t_{m+1}} (LM_j)u_i dx\right] = 0 . \tag{5.8}$$

(iii) $t_{j+k} > t_{n-1}$ and $t_j < t_n$. Thus $t_j \le t_{n-1} < t_n \le t_{j+k}$ must occur. Let

$$\sum_{t_j \le t_m < t_{m+1} \le t_{j+k}} [W(M_j,f;t_m+) - W(M_j,f;t_{m+1}-)] = \Sigma_1 + \Sigma_2 + \Sigma_3 , \tag{5.9}$$

where

$$\Sigma_1 := \sum_{t_j \le t_m < t_{m+1} \le t_{n-1}} [W(M_j,f;t_m+) - W(M_j,f,t_{m+1}-)] + W(M_j,f;t_{n-1}+) , \tag{5.10}$$

$$\Sigma_2 := -W(M_j,f-u_i;t_n-) + \sum_{t_n \le t_m < t_{m+1} \le t_{j+k}} [W(M_j,f-u_i;t_m+) - W(M_j,f-u_i;t_{m+1}-)] , \tag{5.11}$$

$$\Sigma_3 := -W(M_j,u_i;t_n-) + \sum_{t_n \le t_m < t_{m+1} \le t_{j+k}} [W(M_j,u_i;t_m+) - W(M_j,u_i;t_{m+1}-)] . \tag{5.12}$$

It follows from (5.3), (5.4), (5.6) and (5.7) that

$$\Sigma_1 = 0, \quad \Sigma_2 = 0 .$$

A demonstration similar to that in (5.8) gives

$$\sum_{t_n \le t_m < t_{m+1} \le t_{j+k}} [W(M_j, u_i; t_m+) - W(M_j, u_i; t_{m+1}-)] = 0 .$$

Finally we have

$$\int (L^* f) M_j dx = \Sigma_1 + \Sigma_2 + \Sigma_3 = -W(M_j, u_i; t_{n+1}-) = -\delta_{ij} ,$$

that is,

$$\int h_i M_j = \delta_{ij} .$$

This completes the proof of "if" part.

The proof of "only if" part is based on the following lemma.

<u>Lemma 5.1</u>. (1°) If $f^{(\ell_s)}(t_s) = 0$ $(s = j, j+1, \ldots, j+\ell_j)$ and $W(M_{j-1}, f; t_{j-1}+) = 0$, then $f^{(\ell_{j-1})}(t_{j-1}) = 0$.

(2°) If $f^{(c_s)}(t_s) = 0$ $(s = j, j-1, \ldots, j-c_j)$ and $W(M_{j+1}, f; t_{j+1}) = 0$, then $f^{(c_{j+1})}(t_{j+1}) = 0$.

<u>Proof</u>. It suffices to prove (1°), because the proof of (2°) is similar. There are two possibilities.

(i) $t_{j-1} < t_j$. In this case,

$$M_{j-1}(t_{j-1}) = M'_{j-1}(t_{j-1}) = \cdots = M_{j-1}^{(k-2)}(t_{j-1}) = 0, \quad M_{j-1}^{(k-1)}(t_{j-1}+) \neq 0 ,$$

so by (2.5) we have $M_{j-1}^{(k-1)}(t_{j-1}+) f(t_{j-1}) = W(M_{j-1}, f; t_{j-1}+) = 0$, hence $f(t_{j-1}) = 0$.

(ii) $t_{j-1} = t_j$. Putting

$$M_{j-1}(t_{j-1}) = M'_{j-1}(t_{j-1}) = \cdots = M_{j-1}^{(k-\ell_{j-1}-2)}(t_{j-1}) = 0, \quad M_{j-1}^{(k-\ell_j-1)}(t_{j-1}) \neq 0$$

and

$$f(t_{j-1}) = \cdots = f^{(\ell_{j-1}-1)}(t_{j-1}) = 0$$

in the place of the expression (2.5) for $W(M_{j-1}, f; t_{j-1}+)$, we obtain $f^{(\ell_{j-1})}(t_{j-1}) = 0$.

Now we proceed with the proof of the necessity. If $h_i \in L_q(a,b)$ is such a function that $\int h_i M_j = \delta_{ij}$, all j, then there exists a $f \in H_q^k(a,b)$ such that $-L^* f = h_i$ and

$$f^{(\ell_s)}(t_s) = 0, \qquad s = i, i+1, \ldots, n-1 ; \tag{5.13}$$

$$f^{(c_s)}(t_s) = u_i^{(c_s)}(t_s), \qquad s = n, \ldots, i+k-1 . \tag{5.14}$$

To prove $f|_{\underline{\underline{t}}} = u_i^+|_{\underline{\underline{t}}}$, that is to prove

$$f^{(\ell_s)}(t_s) = 0 \quad \text{for all} \quad s \le n-1 , \tag{5.15}$$

$$f^{(c_s)}(t_s) = 0 \quad \text{for all} \quad s \ge n , \tag{5.16}$$

we proceed by induction on s. We only need to prove (5.16), because the proof of (5.15) is similar. Suppose (5.16) is true for s such that $n \le s \le j-1$, where $j \ge i+k$. Consider the integral $\int M_{j-k}(L^* f)dx$. Calculate its value by (5.9)-(5.12). It is easily seen that the contribution of Σ_1 is zero, the contribution of Σ_2 is $-W(M_{j-k}, f - u_i; t_j-)$, and the contribution of Σ_3 is $-\delta_{i,j-k}$. On the other hand, $\int M_{j-k}(L^* f)dx = -\int M_{j-k} h_i dx = -\delta_{i,j-k}$, therefore,

$$W(M_{j-k}, f - u_i; t_j-) = 0 .$$

Resorting to Lemma 5.1, we obtain

$$f^{(c_j)}(t_j) = u_i^{(c_j)}(t_j) .$$

This completes the proof of the "only if" part, and so of the theorem.

Corollary 5.1. If $[\alpha,\beta] \subseteq [t_i, t_{i+k}]$, and if $f \in H_q^k[\alpha,\beta]$ satisfies the following conditions:

(i) $f^{(\gamma)}(\alpha) = 0, \quad \gamma = 0,1,\ldots,k-1;$

(ii) $f^{(\gamma)}(\beta) = u_i^{(\gamma)}(\beta), \quad \gamma = 0,1,\ldots,k-1;$

(iii) $f^{(\gamma)}(t_j) = 0, \quad \gamma = 0,1,\ldots,k-d_j-1$ for $t_j \in (\alpha,\beta);$

then h_i determined by $h_i = -L^* f$ has support $[\alpha,\beta]$ and

$$\int h_i M_j = \delta_{ij} \quad \text{for all} \quad j .$$

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REFERENCES

[1] C. de Boor, Splines as linear combinations of B-splines, in "Approximation Theory II", G. G. Lorentz ed., Academic Press, New York, 1976, 1-47.

[2] C. de Boor, On local linear functionals which vanish at all B-splines but one, in Theory of Approximation with Applications, A. Law and A. Sahney, eds., Academic Press, New York, 1976, 120-145.

[3] C. de Boor, The quasi-interpolant as a tool in elementary polynomial spline theory, in "Approximation Theory", G. G. Lorentz ed., Academic Press, New York, 1973, 269-276.

[4] C. de Boor and G. Fix, Spline approximation by quasi-interpolants, J. Approximation Theory 7 (1973), 19-45.

[5] J. W. Jerome and L. L. Schumaker, Local support bases for a class of spline functions, J. Approximation Theory 16 (1976), 16-27.

[6] T. Muir, A treatise on the theory of determinants, Dover, New York, 1960.

[7] M. A. Naimark, Linear Differential Operators, Part One, translated by E. R. Dawson, F. Ungar Pub. Co., New York, 1967, 6-7.

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Quasi-interpolant functionals for L-splines are constructed. With them as a tool, an explicit construction of LB-splines is done, and a quick proof of the existence and uniqueness of the expansion of an L-spline in an LB-spline series is given. Moreover, a necessary and sufficient condition for a function, under which it generates a local linear functional that vanishes at all LB-splines but one, is obtained.